

Noise as a Boolean algebra of σ -fields. I. Completion

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Abstract

Nonclassical noises over the plane (such as the black noise of percolation) consist of σ -fields corresponding to some planar domains. One can treat less regular domains as limits of more regular domains, thus extending the noise and its set of σ -fields. The greatest extension is investigated in a new general framework.

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Introduction

A noise is defined as a family of σ -fields (in other words, σ -algebras) contained in the σ -field of a probability space and satisfying some conditions. Initially, these σ -fields were indexed by intervals of the real line (the time axis). Recently, a spectacular progress in understanding the full scaling limit

of the critical planar percolation [1] have lead to an important noise over the plane (the black noise of percolation). Its σ -fields are indexed by planar domains with finite-length boundary; “the regularity assumption can be considerably weakened (though it cannot be dropped)” [1, Remark 1.8]. The needed regularity of the domains depends on some properties of the noise. For a classical (white, Poisson or their combination) noise over \mathbb{R}^n , the family of σ -fields extends naturally from nice domains to arbitrary Lebesgue measurable subsets of \mathbb{R}^n . What happens to a nonclassical (in particular, black) noise? One may hope that it extends naturally to the greatest class of subsets of \mathbb{R}^n acceptable for the given noise. For now, nothing like that is proved, nor even conjectured.

It is worth to split the problem in two:

(a) enlarge the given set of σ -fields (irrespective of their relation to the domains in \mathbb{R}^n);

(b) extend the given correspondence between the domains and the σ -fields.

Only the former problem, (a), is treated in this work.

A noise over \mathbb{R} extends readily from intervals to their finite unions, which leads to a lattice homomorphism from the Boolean algebra A of finite unions of intervals modulo finite sets to the lattice Λ of all sub- σ -fields¹ of the σ -field \mathcal{F} of a given probability space (Ω, \mathcal{F}, P) :

$$\mathcal{F}_{a \cap b} = \mathcal{F}_a \cap \mathcal{F}_b, \quad \mathcal{F}_{a \cup b} = \mathcal{F}_a \vee \mathcal{F}_b \quad \text{for } a, b \in A;$$

here $\mathcal{F}_a \vee \mathcal{F}_b$ is the least σ -field containing both \mathcal{F}_a and \mathcal{F}_b . The image $B = \{\mathcal{F}_a : a \in A\}$ is necessarily a sublattice of Λ and a Boolean algebra² such that for all $a, b \in A$,

$$\text{if } \mathcal{F}_a \cap \mathcal{F}_b = \mathcal{F}_\emptyset \quad \text{then } \mathcal{F}_a, \mathcal{F}_b \text{ are independent}$$

(that is, $P(X \cap Y) = P(X)P(Y)$ for all $X \in \mathcal{F}_a, Y \in \mathcal{F}_b$). Thus B is an example to the following definition.

We define a *noise-type Boolean algebra* (of σ -fields) as a sublattice B of Λ , containing the trivial σ -field (only null sets and their complements) and the whole \mathcal{F} , such that B is a Boolean algebra, and any two σ -fields of B are independent whenever their intersection is the trivial σ -field.³

For the black noise of percolation we may start with the Boolean algebra A_1 of finite unions of 2-dimensional intervals $(s_1, t_1) \times (s_2, t_2) \subset \mathbb{R}^2$ modulo finite unions of horizontal and vertical straight lines; or alternatively, the

¹Each σ -field is assumed to contain all null sets.

²That is, Boolean lattice. I do not write “Boolean sublattice” because the lattice of all σ -fields is not Boolean.

³Homogeneity (that is, shift invariance) of a noise is ignored here.

Boolean algebra A_2 of all sets with finite-length boundary, modulo finite-length sets; $A_1 \subset A_2$. We get two noise-type Boolean algebras, $B_1 \subset B_2$. For every $a \in A_2$ there exist $a_n \in A_1$ such that $a_n \uparrow a$ and therefore $\mathcal{F}_{a_n} \uparrow \mathcal{F}_a$; thus, the pair B_1, B_2 is an example to the following definition.

Let $B_1 \subset B_2 \subset \Lambda$ be two noise-type Boolean algebras. We say that B_1 is *monotonically dense* in B_2 if every monotonically closed subset of Λ containing B_1 contains also B_2 . Here a subset $Z \subset \Lambda$ is called monotonically closed if, first, Z contains $\cap_n \mathcal{E}_n$ for every decreasing sequence of σ -fields $\mathcal{E}_n \in Z$, and second, Z contains $\vee_n \mathcal{E}_n$ (the least σ -field containing all \mathcal{E}_n) for every increasing sequence of σ -fields $\mathcal{E}_n \in Z$.

We define the *noise-type completion* of a noise-type Boolean algebra $B \subset \Lambda$ as the greatest among all noise-type Boolean algebras $C \subset \Lambda$ such that $B \subset C$ and B is monotonically dense in C .

It appears that the greatest among these C exists and can be described explicitly.

Theorem 1. Every noise-type Boolean algebra has the noise-type completion.

Theorem 2. Let $B \subset \Lambda$ be a noise-type Boolean algebra. Denote by C its noise-type completion, and by \tilde{B} the least monotonically closed subset of Λ containing B . Then an arbitrary σ -field $\mathcal{E} \in \Lambda$ belongs to C if and only if it satisfies the following two conditions:

- (a) $\mathcal{E} \in \tilde{B}$;
- (b) \mathcal{E} has a complement \mathcal{E}' in \tilde{B} ; that is, $\mathcal{E}' \in \tilde{B}$, $\mathcal{E} \cap \mathcal{E}'$ is the trivial σ -field, and $\mathcal{E} \vee \mathcal{E}'$ is the whole \mathcal{F} (such \mathcal{E}' is necessarily unique).

Note that C is uniquely determined by \tilde{B} .

1 The complete lattice of σ -fields

1a Preliminaries: type L_2 subspaces

Let (Ω, \mathcal{F}, P) be a probability space, and $H = L_2(\Omega, \mathcal{F}, P)$ the corresponding Hilbert space, assumed to be separable. The following two conditions on a (closed linear) subspace H_1 of H are equivalent [2, Th. 3]:

- (a) H_1 is a sublattice of H , containing constants. That is, H_1 contains $f \vee g$ and $f \wedge g$ for all $f, g \in H_1$, where $(f \vee g)(\omega) = \max(f(\omega), g(\omega))$ and $(f \wedge g)(\omega) = \min(f(\omega), g(\omega))$; and H_1 contains the one-dimensional space of constant functions.

- (b) There exists a sub- σ -field $\mathcal{F}_1 \subset \mathcal{F}$ such that $H_1 = L_2(\mathcal{F}_1)$, the space of all \mathcal{F}_1 -measurable functions of H .

Such subspaces H_1 will be called type L_2 (sub)spaces. (In [2] they are called measurable, which can be confusing.)

Each sub- σ -field $\mathcal{F}_1 \subset \mathcal{F}$ is assumed to contain all null sets. Then, the relation $H_1 = L_2(\mathcal{F}_1)$ establishes a bijective correspondence between type L_2 subspaces of H and sub- σ -fields of \mathcal{F} . This correspondence is evidently isotone,

$$H_1 \subset H_2 \quad \text{if and only if} \quad \mathcal{F}_1 \subset \mathcal{F}_2.$$

Thus, we may define a partially ordered set $\Lambda = \Lambda(\Omega, \mathcal{F}, P)$ in two equivalent ways: as consisting of all type L_2 subspaces of H , or alternatively, of all sub- σ -fields of \mathcal{F} ; up to isomorphism, it is the same Λ . An element $x \in \Lambda$ may be thought of as a type L_2 subspace $H_x \subset H$ or a sub- σ -field $\mathcal{F}_x \subset \mathcal{F}$; $H_x = L_2(\mathcal{F}_x)$.

The set Λ contains the greatest element 1 (the whole H , or the whole \mathcal{F}) and the least element 0 (the one-dimensional space of constants, or the trivial σ -field, — only null sets and their complements).

The infimum exists for every subset of Λ , since the intersection of type L_2 spaces is a type L_2 space; alternatively, the intersection of σ -fields is a σ -field.

Existence of the supremum follows readily [3, Th. 2.31]. It is the type L_2 space generated by the union of the given type L_2 spaces. Alternatively, it is the σ -field generated by the union of the given σ -fields. (See [2, Th. 2].)

Thus, Λ is a complete lattice. For two elements $x, y \in \Lambda$ their infimum is denoted by $x \wedge y$, and supremum by $x \vee y$.

1b Bad properties

This subsection is not used in the sequel and may be skipped. Its goal is, to warn the reader against some incorrect arguments that could suggest themselves.

See [3, Sect. 4] about modular and distributive lattices, the diamond M_3 and the pentagon N_5 .

1b1 Remark. The lattice Λ is not modular, and therefore not distributive, unless it is finite.

If (Ω, \mathcal{F}, P) consists of only a finite number n of atoms (and no nonatomic part) then $\dim H = n$. For $n = 3$, $\Lambda = M_3$ is the diamond, modular but not distributive. For $n = 4$, Λ is not modular, since it contains N_5 . Proof: let $\alpha, \beta, \gamma, \delta$ be the four atoms of (Ω, \mathcal{F}, P) , then $\{0, 1, u, v, w\} = N_5$ where $u = \sigma(\alpha, \gamma, \beta \cup \delta)$, $v = \sigma(\alpha \cup \beta, \gamma \cup \delta)$ and $w = \sigma(\alpha \cup \gamma, \beta \cup \delta)$; here $\sigma(\dots)$ is the σ -field generated by (\dots) .

The following two remarks show that the lattice operations, $(x, y) \mapsto x \wedge y$ and $(x, y) \mapsto x \vee y$, generally violate some natural continuity.

1b2 Remark. It may happen that $x_n \uparrow x$ (that is, $x_1 \leq x_2 \leq \dots$ and $\sup_n x_n = x$), $x_n \wedge y = 0$ for all n , but $x \wedge y = y \neq 0$.

Proof. Assuming that (Ω, \mathcal{F}, P) contains $\alpha_1, \alpha_2, \dots \in \mathcal{F}$ such that $\alpha_1 \subset \alpha_2 \subset \dots$, $P(\alpha_1) < P(\alpha_2) < \dots$ and $\lim_n P(\alpha_n) < 1$, we introduce $x_n = \sigma(\alpha_1, \dots, \alpha_n)$, $x = \sigma(\alpha_1, \alpha_2, \dots)$ and $y = \sigma(\alpha)$ where $\alpha = \cup_n \alpha_n$. Then $x_n \uparrow x$; $x_n \wedge y = 0$ (just because $\alpha \notin \sigma(\alpha_1, \dots, \alpha_n)$); and $x \wedge y = y \neq 0$ (since $\alpha \in \sigma(\alpha_1, \alpha_2, \dots)$ and $0 < P(\alpha) < 1$).

1b3 Remark. It may happen that $x_n \downarrow 0$, $x_n \vee y = 1$ for all n , but $y \neq 1$.

Proof. Assuming that (Ω, \mathcal{F}, P) is the interval $[0, 1] \subset \mathbb{R}$ with Lebesgue measure, we introduce the σ -field \mathcal{G} of all measurable sets invariant under the transformation $\omega \mapsto 1 - \omega$, and (for each n) the σ -field \mathcal{F}_n of all measurable sets invariant under the transformation $\omega \mapsto \omega + 2^{-n} \pmod{1}$. The rest is left to the reader.

See also Remark 2a7.

1c More preliminaries: tensor products

The product of two probability spaces leads to the tensor product of Hilbert spaces (assumed to be separable, as before), see for instance [4, Sect. II.4, Th. II.10]; that is,

$$L_2((\Omega', \mathcal{F}', P') \times (\Omega'', \mathcal{F}'', P'')) = L_2(\Omega', \mathcal{F}', P') \otimes L_2(\Omega'', \mathcal{F}'', P'')$$

in the following sense: the formula $(f \otimes g)(\omega', \omega'') = f(\omega')g(\omega'')$ establishes a unitary operator between these two Hilbert spaces.

The same situation appears when two sub- σ -fields $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$ are independent. The formula

$$(f \otimes g)(\omega) = f(\omega)g(\omega) \quad \text{for } f \in L_2(\mathcal{F}_1), g \in L_2(\mathcal{F}_2), \omega \in \Omega$$

establishes a unitary operator from $L_2(\mathcal{F}_1) \otimes L_2(\mathcal{F}_2)$ onto $L_2(\mathcal{F}_1 \vee \mathcal{F}_2)$. This operator is the composition of the operator $L_2(\mathcal{F}_1) \otimes L_2(\mathcal{F}_2) \rightarrow L_2((\Omega, \mathcal{F}_1, P) \times (\Omega, \mathcal{F}_2, P))$ discussed before, and the operator $L_2((\Omega, \mathcal{F}_1, P) \times (\Omega, \mathcal{F}_2, P)) \rightarrow L_2(\Omega, \mathcal{F}_1 \vee \mathcal{F}_2, P)$ conjugated to the measure preserving “diagonal” map

$$(\Omega, \mathcal{F}_1 \vee \mathcal{F}_2, P) \ni \omega \mapsto (\omega, \omega) \in (\Omega, \mathcal{F}_1, P) \times (\Omega, \mathcal{F}_2, P).$$

We need also the equality

$$(1c1) \quad (H_{1a} \otimes H_{2a}) \cap (H_{1b} \otimes H_{2b}) = (H_{1a} \cap H_{1b}) \otimes (H_{2a} \cap H_{2b})$$

for arbitrary (closed linear) subspaces $H_{1a}, H_{1b} \subset H_1$ and $H_{2a}, H_{2b} \subset H_2$ of Hilbert spaces H_1, H_2 . Formula (1c1) follows easily from its special case

$$(H_{1a} \otimes H_2) \cap (H_1 \otimes H_{2a}) = H_{1a} \otimes H_{2a}$$

for $H_{1a} \subset H_1, H_{2a} \subset H_2$. This special case follows from the evident equality

$$(Q_1 \otimes \mathbb{1})(\mathbb{1} \otimes Q_2) = Q_1 \otimes Q_2 = (\mathbb{1} \otimes Q_2)(Q_1 \otimes \mathbb{1}),$$

since for two commuting projections, the image of their product is the intersection of their images.

1d Good properties

Elements $x \in \Lambda$ may be treated as sub- σ -fields $\mathcal{F}_x \subset \mathcal{F}$ or type L_2 subspaces $H_x \subset H = L_2(\Omega, \mathcal{F}, P)$, but also as the corresponding orthogonal projections $Q_x : H \rightarrow H, Q_x H = H_x$, which gives us some useful structures on Λ not derivable from the partial order.

The strong operator topology on the projection operators Q_x gives us a topology on Λ ; we call it the strong operator topology on Λ . It is metrizable (since the strong operator topology is metrizable on operators of norm ≤ 1). Thus,

$$x_n \rightarrow x \quad \text{means} \quad \forall \psi \in H \quad \|Q_{x_n} \psi - Q_x \psi\| \rightarrow 0.$$

Below, “topologically” means “according to the strong operator topology”.

On the other hand we have the monotone convergence derived from the partial order on Λ :

$$\begin{aligned} x_n \downarrow x \quad \text{means} \quad x_1 \geq x_2 \geq \dots \quad \text{and} \quad \inf_n x_n = x, \\ x_n \uparrow x \quad \text{means} \quad x_1 \leq x_2 \leq \dots \quad \text{and} \quad \sup_n x_n = x. \end{aligned}$$

1d1 Definition. (a) A set $Z \subset \Lambda$ is *monotonically closed*, if for all $x_n \in Z$ and $x \in \Lambda$

$$\begin{aligned} x_n \downarrow x \quad \text{implies} \quad x \in Z, \\ x_n \uparrow x \quad \text{implies} \quad x \in Z. \end{aligned}$$

(b) Given two subsets $Z_1 \subset Z_2 \subset \Lambda$, we say that Z_1 is *monotonically dense* in Z_2 if every monotonically closed set containing Z_1 contains also Z_2 .

1d2 Lemma. (a) $x_n \downarrow x$ implies $x_n \rightarrow x$; also, $x_n \uparrow x$ implies $x_n \rightarrow x$;

(b) every set closed in the strong operator topology is monotonically closed.

Proof. Clearly, (b) follows from (a); we have to prove (a).

First, if $x_n \downarrow x$ then $H_{x_1} \supset H_{x_2} \supset \dots$ and $\bigcap_n H_{x_n} = H_x$, therefore $Q_{x_n} \rightarrow Q_x$ strongly. Second, if $x_n \uparrow x$ then $H_{x_1} \subset H_{x_2} \subset \dots$; the closure of $\bigcup_n H_{x_n}$, being a type L_2 space, is equal to H_x ; therefore $Q_{x_n} \rightarrow Q_x$ strongly. \square

1d3 Definition. Elements $x, y \in \Lambda$ are *commuting*,¹ if $Q_x Q_y = Q_y Q_x$. A subset of Λ is *commutative*, if its elements are pairwise commuting.

Note that

(1d4) the topological closure of a commutative set is commutative,

(1d5) if $x, y \in \Lambda$ are commuting then $Q_x Q_y = Q_{x \wedge y}$.

1d6 Proposition. Let

$$B \subset C \subset \Lambda, \quad B \text{ is commutative, } \forall x, y \in B \quad x \wedge y \in B.$$

Then B is monotonically dense in C if and only if B is topologically dense in C (that is, C is contained in the topological closure of B).

The “only if” part follows from 1d2(b). The proof of the “if” part is given after a lemma.

Given $x_n \in \Lambda$, we define

$$\liminf_n x_n = \sup_n \inf_k x_{n+k}, \quad \limsup_n x_n = \inf_n \sup_k x_{n+k}.$$

1d7 Lemma. If $x_n \in \Lambda$ are pairwise commuting and $x_n \rightarrow x$ then

$$\liminf_k x_{n_k} = x$$

for some $n_1 < n_2 < \dots$

Proof. The commuting projection operators Q_{x_n} generate a commutative von Neumann algebra; such algebra is always isomorphic to the algebra L_∞ on some measure space (of finite measure), see for instance [6, Th. 1.22]. Denoting the isomorphism by α we have $\alpha(Q_{x_n}) = \mathbb{1}_{E_n}$, $\alpha(Q_x) = \mathbb{1}_E$ (indicators of measurable sets E_n, E). By (1d5),

$$\alpha(Q_{x_m \wedge x_n}) = \mathbb{1}_{E_m \cap E_n}$$

for all m, n ; the same holds for more than two indices.

¹Not to be confused with the notion mentioned in [5, Chap. II, Sect. 14, p. 52].

The strong convergence of operators $Q_{x_n} \rightarrow Q_x$ implies convergence in measure of indicators, $\mathbb{1}_{E_n} \rightarrow \mathbb{1}_E$ (since $\mathbb{1}_{E_n} = Q_{x_n} \mathbb{1} \rightarrow Q_x \mathbb{1} = \mathbb{1}_E$). We choose a subsequence convergent almost everywhere, $\mathbb{1}_{E_{n_k}} \rightarrow \mathbb{1}_E$, then $\liminf_k \mathbb{1}_{E_{n_k}} = \mathbb{1}_E$, that is,

$$\sup_k \inf_i \mathbb{1}_{E_{n_k+i}} = \mathbb{1}_E.$$

We have $\alpha(Q_{x_{n_k} \wedge x_{n_{k+1}} \wedge \dots \wedge x_{n_{k+i}}}) = \mathbb{1}_{E_{n_k} \cap E_{n_{k+1}} \cap \dots \cap E_{n_{k+i}}}$, therefore (for $i \rightarrow \infty$, using 1d2(a)), $\alpha(Q_{\inf_i x_{n_{k+i}}}) = \inf_i \mathbb{1}_{E_{n_{k+i}}}$, and further (for $k \rightarrow \infty$), $\alpha(Q_{\sup_k \inf_i x_{n_{k+i}}}) = \sup_k \inf_i \mathbb{1}_{E_{n_{k+i}}}$. We get $\alpha(Q_{\liminf_k x_{n_k}}) = \liminf_k \mathbb{1}_{E_{n_k}} = \mathbb{1}_E = \alpha(Q_x)$, therefore $\liminf_k x_{n_k} = x$. \square

Proof of Proposition 1d6, the “if” part. Let Z be a monotonically closed set, $Z \supset B$, and $x \in C$; we have to prove that $x \in Z$.

There exist $x_n \in B$ such that $x_n \rightarrow x$. By 1d7 we may assume that $\liminf_n x_n = x$. We have $x_n \wedge x_{n+1} \wedge \dots \wedge x_{n+k} \in B \subset Z$ for all k and n , which implies $\inf_k x_{n+k} \in Z$ and further $x = \sup_n \inf_k x_{n+k} \in Z$. \square

It will be shown (see 2a2) that every noise-type Boolean algebra B is a commutative subset of Λ . Thus, by Proposition 1d6, the following definition is equivalent to that of the introduction.

1d8 Definition. The *noise-type completion* of a noise-type Boolean algebra $B \subset \Lambda$ is the greatest among all noise-type Boolean algebras $C \subset \Lambda$ such that $B \subset C$ and B is topologically dense in C (according to the strong operator topology).

Also, by 1d6, the least monotonically closed subset of Λ containing B is equal to the topological closure of B .

1d9 Corollary. In Theorem 2, the set \tilde{B} may be replaced with the topological closure of B (according to the strong operator topology).

Thus, the “monotonical” notions are eliminated.

FROM NOW ON, CONVERGENCE, DENSENESS AND CLOSENESS ARE ALWAYS TOPOLOGICAL (ACCORDING TO THE STRONG OPERATOR TOPOLOGY). NO OTHER TOPOLOGY ON Λ WILL BE USED.

1d10 Proposition. Let $x_n, y_n, x, y \in \Lambda$, $x_n \rightarrow x$, $y_n \rightarrow y$, and for each n (separately), x_n, y_n commute. Then $x_n \wedge y_n \rightarrow x \wedge y$.

Proof. Note that $Q_{x_n}Q_{y_n} \rightarrow Q_xQ_y$, since for each $\psi \in H$,

$$\|Q_{x_n}Q_{y_n}\psi - Q_xQ_y\psi\| \leq \|Q_{x_n}\| \cdot \|(Q_{y_n} - Q_y)\psi\| + \|(Q_{x_n} - Q_x)Q_y\psi\| \rightarrow 0.$$

Similarly, $Q_{y_n}Q_{x_n} \rightarrow Q_yQ_x$. We have $Q_{x_n}Q_{y_n} = Q_{y_n}Q_{x_n}$, therefore $Q_xQ_y = Q_yQ_x$. By (1d5), $Q_{x \wedge y} = Q_xQ_y$. Similarly, $Q_{x_n \wedge y_n} = Q_{x_n}Q_{y_n}$. We get $Q_{x_n \wedge y_n} \rightarrow Q_{x \wedge y}$, that is, $x_n \wedge y_n \rightarrow x \wedge y$. \square

1d11 Definition. Elements $x, y \in \Lambda$ are *independent*, if the corresponding σ -fields $\mathcal{F}_x, \mathcal{F}_y$ are independent.

It means, $P(X \cap Y) = P(X)P(Y)$ for all $X \in \mathcal{F}_x, Y \in \mathcal{F}_y$. Or equivalently, $\langle Q_x\xi, Q_y\psi \rangle = \langle Q_x\xi, \mathbb{1} \rangle \langle Q_y\psi, \mathbb{1} \rangle$ for all $\xi, \psi \in H$.

1d12 Proposition. The following two conditions on $x, y \in \Lambda$ are equivalent:

- (a) x, y are independent;
- (b) x, y commute, and $x \wedge y = 0$.

Proof. (a) \implies (b): independence of $\mathcal{F}_x, \mathcal{F}_y$ implies $\mathbb{E}(f | \mathcal{F}_y) = \mathbb{E}f$ for all $f \in L_2(\mathcal{F}_x)$; that is, $Q_yf = \langle f, \mathbb{1} \rangle \mathbb{1}$ for $f \in H_x$, and therefore $Q_yQ_x = Q_0 = Q_xQ_y$.

(b) \implies (a): by (1d5), $Q_yQ_x = Q_0 = Q_xQ_y$; thus $Q_yf = \langle f, \mathbb{1} \rangle \mathbb{1}$ for $f \in H_x$, and therefore $\mathbb{P}(A \cap B) = \langle \mathbb{1}_A, \mathbb{1}_B \rangle = \langle \mathbb{1}_A, Q_y\mathbb{1}_B \rangle = \langle Q_y\mathbb{1}_A, \mathbb{1}_B \rangle = \langle \mathbb{1}_A, \mathbb{1} \rangle \langle \mathbb{1}, \mathbb{1}_B \rangle = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{F}_x, B \in \mathcal{F}_y$. \square

It follows that all pairs $(x, y) \in \Lambda \times \Lambda$ such that x, y are independent are a closed set in $\Lambda \times \Lambda$ (in the product topology).

It may happen that $x \wedge y = 0$ but x, y do not commute.¹

For every $x \in \Lambda$ the triple $(\Omega, \mathcal{F}_x, P|_{\mathcal{F}_x})$ is also a probability space, and it may be used similarly to (Ω, \mathcal{F}, P) , giving the complete lattice $\Lambda(\mathcal{F}_x) = \Lambda(\Omega, \mathcal{F}_x, P|_{\mathcal{F}_x})$ endowed with the topology, etc. The evident lattice isomorphism

$$\Lambda_x = \{y \in \Lambda : y \leq x\} \cong \Lambda(\mathcal{F}_x)$$

is also a homeomorphism. Proof: if $y \leq x$ then $H_y \subset H_x \subset H$ and therefore $Q_y = Q_y^{(x)}Q_x$ where $Q_y^{(x)} : H_x \rightarrow H_x$ is the orthogonal projection onto H_y . It follows that $Q_{y_n} \rightarrow Q_y$ if and only if $Q_{y_n}^{(x)} \rightarrow Q_y^{(x)}$. That is, $y_n \rightarrow y$ in Λ_x if and only if $y_n \rightarrow y$ in $\Lambda(\Omega, \mathcal{F}_x, P)$.

Given $x, y \in \Lambda$, the product set $\Lambda_x \times \Lambda_y$ carries the product topology and the product partial order, and is again a lattice (see [3, Sect. 2.15] for the product of two lattices), moreover, a complete lattice (see [3, Exercise 2.26(ii)]).

¹For example, v and w of 1b1 are independent if and only if $P(\alpha)P(\delta) = P(\beta)P(\gamma)$; u and w are never independent.

1d13 Proposition. If $x, y \in \Lambda$ are independent then the map

$$\Lambda_x \times \Lambda_y \ni (u, v) \mapsto u \vee v \in \Lambda_{x \vee y}$$

is an embedding, both algebraically and topologically. In other words, this map is both a lattice isomorphism and a homeomorphism between $\Lambda_x \times \Lambda_y$ and its image $\Lambda_{x,y} = \{u \vee v : u \in \Lambda_x, v \in \Lambda_y\}$ treated as a sublattice and a topological subspace of $\Lambda_{x \vee y}$.

1d14 Remark. If $x \wedge y = 0$ but x, y are not independent then the map need not be one-to-one. For example, 1b1 gives us u, v, w such that $u \wedge v = 0$, $w < u$ and $w \vee v = 1$. Thus, the map $\Lambda_u \times \Lambda_v \rightarrow \Lambda$ sends to 1 both (u, v) and (w, v) .

Proof of Proposition 1d13. According to Sect. 1c, independence of x, y implies

$$H_{x \vee y} = H_x \otimes H_y, \quad (f \otimes g)(\cdot) = f(\cdot)g(\cdot) \quad \text{for } f \in H_x, g \in H_y.$$

We may treat Λ_x as consisting of all L_2 -type subspaces $H_u \subset H_x$, or the corresponding projections $Q_u : H_x \rightarrow H_x$. The same holds for Λ_y and $\Lambda_{x \vee y}$.

Treating $H_x \otimes H_y$ as $H_{x \vee y}$ we get $H_u \otimes H_v = H_{u \vee v} \subset H_{x \vee y}$, thus, $Q_u \otimes Q_v = Q_{u \vee v}$ for $u \in \Lambda_x, v \in \Lambda_y$. If $u_n \rightarrow u$ and $v_n \rightarrow v$ then $u_n \vee v_n \rightarrow u \vee v$, since $Q_{u_n \vee v_n} = Q_{u_n} \otimes Q_{v_n} \rightarrow Q_u \otimes Q_v$. It means that the map $J : \Lambda_x \times \Lambda_y \rightarrow \Lambda_{x \vee y}$, $J(u, v) = u \vee v$, is continuous. This map preserves lattice operations, that is,

$$(1d15) \quad \begin{aligned} (u_1 \vee v_1) \vee (u_2 \vee v_2) &= (u_1 \vee u_2) \vee (v_1 \vee v_2), \\ (u_1 \vee v_1) \wedge (u_2 \vee v_2) &= (u_1 \wedge u_2) \vee (v_1 \wedge v_2) \end{aligned}$$

for all $u_1, u_2 \in \Lambda_x, v_1, v_2 \in \Lambda_y$. The former equality is trivial; the latter equality follows from (1c1) applied to $H_1 = H_x, H_2 = H_y, H_{1a} = H_{u_1}, H_{2a} = H_{v_1}, H_{1b} = H_{u_2}, H_{2b} = H_{v_2}$. We have to prove that J is one-to-one and the inverse map is continuous.

By (1d15), $(u \vee v) \wedge x = u$ and $(u \vee v) \wedge y = v$ for all $u \in \Lambda_x, v \in \Lambda_y$. Thus J is one-to-one. It remains to prove that the maps $z \mapsto z \wedge x, z \mapsto z \wedge y$ are continuous on $J(\Lambda_x \times \Lambda_y)$. Let $z_n, z \in J(\Lambda_x \times \Lambda_y), z_n \rightarrow z$. We introduce $u_n = z_n \wedge x, u = z \wedge x, v_n = z_n \wedge y, v = z \wedge y$, then $u_n, u \in \Lambda_x, v_n, v \in \Lambda_y, u_n \vee v_n = z_n$ and $u \vee v = z$. We note that $(Q_{u_n} \otimes Q_{v_n})(Q_x \otimes Q_0) = Q_{u_n} \otimes Q_0 = (Q_x \otimes Q_0)(Q_{u_n} \otimes Q_{v_n})$, that is, z_n, x commute (for each n separately). By 1d10, $z_n \wedge x \rightarrow z \wedge x$, that is, $u_n \rightarrow u$. Similarly, $v_n \rightarrow v$. \square

1d16 Remark. We see that $\Lambda_x \times \Lambda_y$ (for independent $x, y \in \Lambda$) is naturally isomorphic to the sublattice

$$\Lambda_{x,y} = \{u \vee v : u \in \Lambda_x, v \in \Lambda_y\} = \{u \vee v : u \leq x, v \leq y\} \subset \Lambda.$$

The correspondence between a pair $(u, v) \in \Lambda_x \times \Lambda_y$ and $z \in \Lambda_{x,y}$ is given by

$$\begin{aligned} z &= u \vee v, \\ u &= z \wedge x, \quad v = z \wedge y. \end{aligned}$$

Therefore

$$(1d17) \quad \Lambda_{x,y} = \{z \in \Lambda : z = (z \wedge x) \vee (z \wedge y)\}.$$

The continuous map

$$\Lambda_{x,y} \ni z \mapsto z \wedge x \in \Lambda_x$$

is a lattice homomorphism (and the same holds for the similar map with y in place of x):

$$(1d18) \quad \forall z_1, z_2 \in \Lambda_{x,y} \quad (z_1 \vee z_2) \wedge x = (z_1 \wedge x) \vee (z_2 \wedge x)$$

and of course, $(z_1 \wedge z_2) \wedge x = (z_1 \wedge x) \wedge (z_2 \wedge x)$. Thus, any relation between elements of $\Lambda_{x,y}$ expressed in terms of lattice operations is equivalent to the conjunction of two similar relations “restricted” to x and y . For example, the relation

$$(z_1 \vee z_2) \wedge z_3 = z_4 \vee z_5$$

between $z_1, z_2, z_3, z_4, z_5 \in \Lambda_{x,y}$ splits in two:

$$((z_1 \wedge x) \vee (z_2 \wedge x)) \wedge (z_3 \wedge x) = (z_4 \wedge x) \vee (z_5 \wedge x)$$

and a similar relation with y in place of x .

2 Noise-type Boolean algebras

2a Distributivity relations

Throughout this section $B \subset \Lambda$ is a noise-type Boolean algebra, as defined below.

2a1 Definition. A noise-type Boolean algebra is a sublattice B of Λ , containing 0 and 1, such that B is a Boolean algebra and all $x, y \in B$ satisfying $x \wedge y = 0$ are independent.

Every $x \in B$ has its complement $x' \in B$;

$$x \wedge x' = 0, \quad x \vee x' = 1.$$

(The complement in B is unique, however, many other complements may exist in Λ .)

2a2 Lemma. ¹ $Q_x Q_y = Q_{x \wedge y}$ for all $x, y \in B$.

Proof. For every $\psi \in H$ of the form $\psi = \psi_{00}\psi_{01}\psi_{10}\psi_{11}$ where $\psi_{11} \in H_{x \wedge y}$, $\psi_{10} \in H_{x \wedge y'}$, $\psi_{01} \in H_{x' \wedge y}$, $\psi_{00} \in H_{x' \wedge y'}$, we have

$$\begin{aligned} Q_y \psi &= Q_y((\psi_{00}\psi_{10})(\psi_{01}\psi_{11})) = \langle \psi_{00}\psi_{10}, \mathbb{1} \rangle \psi_{01}\psi_{11} = \langle \psi_{00}, \mathbb{1} \rangle \langle \psi_{10}, \mathbb{1} \rangle \psi_{01}\psi_{11}; \\ Q_x Q_y \psi &= \langle \psi_{00}, \mathbb{1} \rangle \langle \psi_{10}, \mathbb{1} \rangle \langle \psi_{01}, \mathbb{1} \rangle \psi_{11}; \\ Q_{x \wedge y} \psi &= Q_{x \wedge y}((\psi_{00}\psi_{01}\psi_{10})\psi_{11}) = \langle \psi_{00}\psi_{01}\psi_{10}, \mathbb{1} \rangle \psi_{11} = Q_x Q_y \psi. \end{aligned}$$

It follows that $Q_x Q_y = Q_{x \wedge y}$, since linear combinations of the considered ψ are dense in H . \square

We denote by $\text{Cl}(B)$ the closure² of B ;

$$(2a3) \quad \text{Cl}(B) \text{ is commutative,}^3$$

and for all $x, y \in \text{Cl}(B)$,

$$(2a4) \quad x \wedge y \in \text{Cl}(B),$$

$$(2a5) \quad x \wedge y = 0 \text{ if and only if } x, y \text{ are independent.}^4$$

Proof: by (2a2), B is commutative, which implies (2a3) by (1d4); (2a4) follows from 1d10 and (2a3); (2a5) follows from 1d12 and (2a3).

By 1d10 and (2a3),

$$(2a6) \quad x_n \wedge y_n \rightarrow x \wedge y$$

whenever $x_n, x, y_n, y \in \text{Cl}(B)$, $x_n \rightarrow x$, $y_n \rightarrow y$.

2a7 Remark. Surprisingly, $x_n \vee y_n$ need not converge to $x \vee y$, even if $x_n \in B$, $x_n \downarrow 0$, $y_n = x'_n$; it may happen that $y_n \uparrow y$, $y \neq 1$. “The phenomenon ... tripped up even Kolmogorov and Wiener” [7, p. 48]. Also, it may happen that $x_n \in B$, $x_n \rightarrow 0$, $y_n = x'_n$, and the projectors Q_{y_n} converge weakly (that is, in the weak operator topology) to an operator that is not a projection, and therefore no subsequence of $(x'_n)_n$ converges in Λ .

On the other hand it can be shown that if $x_n \in B$, $x_n \rightarrow 1$, then necessarily $x'_n \rightarrow 0$.

2a8 Lemma.

$$\forall x \in \text{Cl}(B) \quad \forall y, z \in B \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

¹Well-known long ago (in slightly different form).

²Topological, of course; recall the note after 1d9.

³As defined by 1d3.

⁴As defined by 1d11.

Proof. First, consider the case $y \wedge z = 0$. By (2a5), y, z are independent. We take $x_n \in B$ such that $x_n \rightarrow x$. By (2a6), $x_n \wedge y \rightarrow x \wedge y$, $x_n \wedge z \rightarrow x \wedge z$ and $x_n \wedge (y \vee z) \rightarrow x \wedge (y \vee z)$. Applying 1d13 to $(x_n \wedge y, x_n \wedge z) \in \Lambda_y \times \Lambda_z$ we get $(x_n \wedge y) \vee (x_n \wedge z) \rightarrow (x \wedge y) \vee (x \wedge z)$. On the other hand, $(x_n \wedge y) \vee (x_n \wedge z) = x_n \wedge (y \vee z)$ (since B is distributive). Thus, $x_n \wedge (y \vee z) \rightarrow (x \wedge y) \vee (x \wedge z)$ and so, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Second, if $y \wedge z \neq 0$, we introduce $u = y \wedge z'$, $v = y \wedge z$, $w = y' \wedge z$, note that $u \vee v \vee w = y \vee z$, $u \wedge v = 0$, $u \wedge w = 0$, $v \wedge w = 0$, and apply several times the special case proved above:

$$\begin{aligned} x \wedge y &= x \wedge (u \vee v) = (x \wedge u) \vee (x \wedge v); \\ x \wedge z &= x \wedge (v \vee w) = (x \wedge v) \vee (x \wedge w); \\ x \wedge (y \vee z) &= x \wedge (u \vee (v \vee w)) = (x \wedge u) \vee (x \wedge (v \vee w)) = \\ &= (x \wedge u) \vee (x \wedge v) \vee (x \wedge w). \end{aligned}$$

□

Taking $y = z'$ we get $x = (x \wedge z) \vee (x \wedge z')$, that is (recall (1d17)),

$$\forall z \in B \quad \text{Cl}(B) \subset \Lambda_{z, z'}.$$

By (1d18),

$$(2a9) \quad \forall x, y \in \text{Cl}(B) \quad \forall z \in B \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

2a10 Proposition. Let $x, y \in \text{Cl}(B)$, $x \wedge y = 0$, $x \vee y = 1$. Then $\text{Cl}(B) \subset \Lambda_{x, y}$, that is,

$$\forall z \in \text{Cl}(B) \quad z = (x \wedge z) \vee (y \wedge z).$$

Proof. We take $z_n \in B$ such that $z_n \rightarrow z$ and apply (2a9):

$$\forall n \quad z_n = (x \wedge z_n) \vee (y \wedge z_n).$$

By (2a5), x and y are independent. We have $z_n = u_n \vee v_n \in \Lambda_{x, y}$ where

$$u_n = x \wedge z_n \in \Lambda_x, \quad v_n = y \wedge z_n \in \Lambda_y.$$

By (2a6), $u_n \rightarrow u = x \wedge z$ and $v_n \rightarrow v = y \wedge z$. By 1d13, $u_n \vee v_n \rightarrow u \vee v$. On the other hand, $u_n \vee v_n = z_n \rightarrow z$. Thus, $z = u \vee v = (x \wedge z) \vee (y \wedge z)$. □

2a11 Lemma. For every $x \in \text{Cl}(B)$ there exists at most one $y \in \text{Cl}(B)$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Proof. Assume that $y_1, y_2 \in \text{Cl}(B)$, $x \wedge y_k = 0$ and $x \vee y_k = 1$ for $k = 1, 2$. By 2a10, $y_2 \in \Lambda_{x, y_1}$, that is, $y_2 = (x \wedge y_2) \vee (y_1 \wedge y_2) = y_1 \wedge y_2$. Similarly, $y_1 = y_2 \wedge y_1$. □

2b Noise-type completion

Still, $B \subset \Lambda$ is a noise-type Boolean algebra. We define

$$(2b1) \quad C = \{x \in \text{Cl}(B) : \exists y \in \text{Cl}(B) (x \wedge y = 0, x \vee y = 1)\}.$$

For $x \in C$ we denote such y (unique by 2a11) by x' . We have

$$(2b2) \quad B \subset C \subset \text{Cl}(B),$$

and for every $x \in C$,

$$(2b3) \quad x' \in C; \quad (x')' = x;$$

$$(2b4) \quad x \wedge x' = 0, \quad x \vee x' = 1.$$

By (2a5), x, x' are independent; and by 2a10,

$$(2b5) \quad \forall x \in C \quad \text{Cl}(B) \subset \Lambda_{x,x'}.$$

2b6 Lemma. For every $x \in C$ the map

$$\text{Cl}(B) \ni z \mapsto x \vee z \in \Lambda$$

is continuous.

Proof. Let $z_n, z \in \text{Cl}(B)$, $z_n \rightarrow z$; we have to prove that $x \vee z_n \rightarrow x \vee z$. By (2a6), $x' \wedge z_n \rightarrow x' \wedge z$. Applying 1d13 to $(x, x' \wedge z_n) \in \Lambda_x \times \Lambda_{x'}$ we get $x \vee (x' \wedge z_n) \rightarrow x \vee (x' \wedge z)$. It remains to prove that $x \vee (x' \wedge z_n) = x \vee z_n$ and $x \vee (x' \wedge z) = x \vee z$. We prove the latter; the former is similar. We have $z \in \text{Cl}(B) \subset \Lambda_{x,x'}$ by (2b5). According to 1d16 we apply the lattice homomorphisms $\Lambda_{x,x'} \ni y \mapsto y \wedge x \in \Lambda_x$ and $\Lambda_{x,x'} \ni y \mapsto y \wedge x' \in \Lambda_{x'}$ to $y = x \vee z$: $(x \vee z) \wedge x = x$ and $(x \vee z) \wedge x' = (x \wedge x') \vee (z \wedge x') = z \wedge x'$, therefore $x \vee z = x \vee (x' \wedge z)$. \square

2b7 Lemma.

$$\forall x \in C \quad \forall y \in \text{Cl}(B) \quad x \vee y \in \text{Cl}(B).$$

Proof. By 2b6 it is sufficient to consider $y \in B$. Applying 2b6 (again) to $y \in B \subset C$ we see that the map $\text{Cl}(B) \ni z \mapsto y \vee z \in \Lambda$ is continuous. This map sends B into B , therefore it sends $x \in C \subset \text{Cl}(B)$ into $\text{Cl}(B)$. \square

2b8 Lemma. For all $x, y \in C$,

$$x \vee y \in C \quad \text{and} \quad (x \vee y)' = x' \wedge y'.$$

Proof. By 2b7, $x \vee y \in \text{Cl}(B)$. By (2a4), $x' \wedge y' \in \text{Cl}(B)$. We have to prove that $(x \vee y) \wedge (x' \wedge y') = 0$ and $(x \vee y) \vee (x' \wedge y') = 1$. We do it using 1d16 (similarly to the proof of 2b6).

First, $x, y, x', y' \in C \subset \text{Cl}(B) \subset \Lambda_{x,x'}$.

Second, we consider $z = (x \vee y) \wedge (x' \wedge y')$ and get $z \wedge x = (x \vee (y \wedge x)) \wedge (0 \wedge (y' \wedge x)) = 0$. Similarly, $z \wedge x' = (0 \vee (y \wedge x')) \wedge x' \wedge (y' \wedge x') \leq y \wedge y' = 0$. Therefore $z = 0$, that is, $(x \vee y) \wedge (x' \wedge y') = 0$.

Third, we consider $z = (x \vee y) \vee (x' \wedge y')$ and get $z \wedge x = x \vee (y \wedge x) \vee (x' \wedge y' \wedge x) = x$. Similarly, $z \wedge x' = (x \wedge x') \vee (y \wedge x') \vee (x' \wedge y' \wedge x') = (y \wedge x') \vee (y' \wedge x') = (y \vee y') \wedge x' = x'$. Therefore $z = x \vee x' = 1$, that is, $(x \vee y) \vee (x' \wedge y') = 1$. \square

In addition, $(x \vee y)' = x' \wedge y' \in C$; applying it to x', y' we see that $x \wedge y \in C$ for all $x, y \in C$, and so,

(2b9) C is a sublattice of Λ .

The lattice C is distributive by (1d18), since $C \subset \text{Cl}(B) \subset \Lambda_{x,x'}$ for all $x \in C$. Also, $0 \in C$, $1 \in C$, and each $x \in C$ has a complement x' in C . It means that C is a Boolean lattice, that is, a Boolean algebra.

2b10 Proposition. The noise-type completion of B (as defined by 1d8) exists and is equal to C .

Proof. Being a Boolean algebra satisfying (2a5), C is a noise-type Boolean algebra, and $B \subset C \subset \text{Cl}(B)$. We have to prove that C contains any other noise-type Boolean algebra C_1 such that $B \subset C_1 \subset \text{Cl}(B)$. This is easy: each $x \in C_1$ has a complement x' in $C_1 \subset \text{Cl}(B)$, therefore $x \in C$ just by (2b1). \square

Theorem 1 follows immediately. Also Theorem 2 follows, since its conditions are mirrored by the definition of C .

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